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REPORT NO. 1504

GENERALIZATION OF THE SIMILARITY EQUATIONS FOR CONTINUOUS SURFACE BOUNDARY LAYERS

by

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October 1970

U.S. ARMY ABERDEEN RESEARCH AND DEVELOPMENT CENTER
BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

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Exterior Ballistics Laboratory

RDT&E Project No. 1T061102A33D

A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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JEDanberg/lca
Aberdeen Proving Ground, Md.
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ABSTRACT

Through a generalization of the continuous surface problem (i.e., boundary layers on a moving belt or behind a shock wave), an equation is derived which encompasses many of the well known similarity solutions of boundary layer theory as well as some solutions not previously considered. Since these problems involve both free stream and wall velocities, similarity variables are introduced which depend on velocity differences. The parameter $B = U_\infty/(U_w - U_\infty)$ describes the relative importance of the boundary conditions along with the usual pressure gradient parameter, β. This formulation of the problem includes the following special cases: flat plate (Blasius), accelerating or decelerating flow (Falkner-Skan), boundary layer behind a shock or expansion wave (Mirels), continuous surface (Sakiadis) and accelerating wall and free stream (Moore). Two new conditions not previously considered involve reverse flow and acceleration and deceleration of a continuous surface boundary layer. Preliminary numerical calculations have been made for these conditions.

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LIST OF SYMBOLS

c_f	Skin friction coefficient
f	Dimensionless stream function
g	y scale function
m	Velocity gradient parameter
t	Time
x	Coordinate along the surface
y	Coordinate normal to the surface
A	$(U_w + U_\infty) / \lvert(U_w + U_\infty)\rvert = (2B + 1) / \lvert(2B + 1)\rvert$
B	$U_\infty / (U_w - U_\infty)$
M_s	Shock Mach number
Pr	Prandtl number
Re_x	Reynolds number
U	Velocity in x direction
V	Velocity in y direction
$\bar{V}$	Dimensionless y velocity
β	Velocity gradient parameter = $2m / (m + 1)$
γ	Ratio of specific heats
δ	Boundary layer thickness
δ^*	Displacement thickness
$\bar{\delta}^*$	Dimensionless displacement thickness
η	Similarity coordinate = y/g
ν	Kinematic viscosity
θ	Momentum thickness
$\bar{\theta}$	Dimensionless momentum thickness

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ρ Density

τ Shear stress

Subscripts

w Wall

∞ Freestream

I. INTRODUCTION

B. C. Sakiadis[1,2,3*] published a number of papers on the so-called continuous solid surface problem in 1961 in which he considered the two-dimensional and axisymmetric cases. In the simplest two-dimensional situation the problem is posed as follows: consider a smooth flat semi-infinite belt issuing from a slot into a region of initially still but viscous fluid. The moving belt entrains some of the neighboring fluid through the action of viscosity and ultimately a steady state situation results in which a boundary layer develops near the belt, beginning at the slot and extending to infinity in the x direction. (See Figure 1.)

In the original paper, the boundary layer equations were employed without proof that they were applicable; however, the results of the analysis did indicate the existence of a boundary layer like situation with one of the major differences being that the y component of velocity of the external fluid is directed toward the belt whereas in the usual boundary situation, the y component of velocity is directed away from the wall. It was further shown that a similarity approach could be used to reduce the boundary layer equations to the Blasius equation:

$$f''' + f\,f'' = 0 \qquad\qquad f = f(\eta)$$
$$\eta = y/\sqrt{2\nu x/U_w}$$

with boundary conditions different from those of the Blasius problem of a stationary flat plate in a uniform flow. (Reference 1 contained a factor 1/2 in front of f''' but this can be eliminated by a simple change of variable.)

Boundary Conditions

Continuous Surface		Flat Plate (Blasius)	
$f(0) = 0$	$f'(0) = 1$	$f(0) = 0$	$f'(0) = 0$
	$f'(\infty) = 0$		$f'(\infty) = 1$

* *References are listed on page 45.*

The resulting numerical solutions for the continuous surface case were found to be different from the Blasius result despite the fact that the equation is identical. Some of the differences which were found are:

	Continuous Surface	Flat Plate (Blasius)
$\delta/\sqrt{\nu x/U_w}$	6.37	4.91
$c_f\sqrt{Re_x}$	0.880	0.664

The reason for the difference between the flat plate and the continuous surface solution is because the equation is nonlinear and therefore, the boundary conditions cannot be transformed without affecting the equation itself. The physical significance of this mathematical statement will be considered in the next section. In the third section, the relationship between the continuous surface problem of Sakiadis and the analysis of the shock tube boundary layer of Mirels[4,5] is discussed. This leads to a generalization whereby a single equation is derived which covers many of the well known similarity solutions and indicates the possibility of some new results.

In the following, it will be assumed that the boundary layer equations are applicable. It is recognized, however, that a proof is not available. Furthermore, the successful numerical integration of these equations does not definitely establish whether or not solutions, in fact, exist. Consistent with boundary layer theory, it is assumed that a characteristic Reynolds number is large and that the main effects of viscosity are confined to a relatively thin region (compared to the distance along the surface) near the surface.

Consideration will be limited to incompressible, similarity problems. The restriction to similar flows implies that velocity profiles when written in non-dimensional form are a unique function of a non-dimensional coordinate perpendicular to the surface and thus, the equations of motion reduce to ordinary differential equations. Sakiadis[1,2,3] has studied this problem as well as extending his work to include approximate

integral techniques, investigating the nonsimilar axisymmetric and turbulent situations. Fox, et al[6], have analyzed the heat transfer problem on continuous surfaces and Tsou, et al[7], have measured the heat transfer and flow properties of laminar and turbulent continuous surface boundary layers. A number of other investigators[7,8] have also contributed to the knowledge of continuous surfaces including the application to ground plane simulators for wind tunnels[8], to aircraft flaps[9] and to the study of rapidly accelerated boundary layers[10]. Moore[11] has also used two solutions of accelerating wall and freestream flows in a study of unsteady boundary layer separation and it is believed that a more complete investigation of the equations developed here might also contribute to the study of the criteria for boundary layer separation.

II. RELATIONSHIP BETWEEN STEADY AND UNSTEADY BOUNDARY LAYER EQUATIONS

It is an axiom of fluid mechanics that dynamic phenomena only depend on the relative velocity between the fluid and its boundaries and not on any absolute velocities, at least as long as all velocities are small compared with the speed of light. Sakiadis results, however, are numerically different from those obtained using the same boundary layer equations with the only difference being a reversal of the f' boundary conditions. The reason for the difference is, of course, that the same equations do not apply when a uniform velocity is superposed to change the boundary conditions. For example, the flat plate in a uniform flow involves the steady state equations of motion. If a uniform velocity in the opposite direction is imposed to reverse the boundary conditions, it is necessary to use the unsteady equations of motion.

Consider the two coordinate systems of Figure 2. Where the x_1, y coordinates are fixed in space and the plate moves in the positive x_1 direction away from the origin, the x_2, y system, on the other hand, is fixed with respect to the plate and "sees" a free stream flow in the x_2 direction.

The transformation of any x coordinate in the x_2, y coordinate system, into the x_1, y system is given by:

$$x_2 = U_\infty t - x_1 \qquad (1)$$

Differentiation with respect to time gives a relationship between velocities in the two systems.

$$U_2 = U_\infty - U_1 \qquad (2)$$

This last equation must be considered in the sense that $U_2 = U_2(x_2,y)$ and that $U_1 = U_1\ (x_1,y$ and t = constant) where $x_2 = U_\infty t - x_1$. That is, equation (2) relates the steady state velocity field (subscript 2) with the unsteady velocity field U_1 at a given instant in time.

The unsteady momentum equation with consistent boundary layer assumptions is:

$$\frac{\partial U_1}{\partial t} + U_1 \frac{\partial U_1}{\partial x_1} + V_1 \frac{\partial U_1}{\partial y} = \nu \frac{\partial^2 U_1}{\partial y^2} \qquad (3)$$

It will be assumed that a solution of the unsteady momentum equation can be obtained in the form

$$\frac{U_1}{U_\infty} = f_1'(\eta) \text{ where } \eta = y/g(x_1,t)$$

When these assumptions are applied to the unsteady momentum equation, and using the continuity equation, the result is:

$$f_1''' + \frac{g}{\nu} \frac{\partial g}{\partial t} \eta_1 f_1'' + \frac{U_\infty}{\nu} g \frac{\partial g}{\partial x_1} f_1 f_1'' = 0 \qquad (4)$$

In order for equation (4) to have a similarity solution, it must reduce to an ordinary differential equation with constant coefficients, α_1 and α_2 for the terms $\eta_1 \, f_1''$ and $f_1 \, f_1''$.

Thus $\quad \alpha_1 = \frac{g}{\nu} \frac{\partial g}{\partial t} \qquad$ implying $\quad g^2/2 = \nu\alpha_1 t + c_1(x_1)$

and $\quad \alpha_2 = \frac{U_\infty}{\nu} \, g \, \frac{\partial g}{\partial x_1} \qquad$ implying $\quad g^2/2 = \frac{\nu}{U_\infty} \, \alpha_2 x_1 + c_2(t)$

thus $\quad g^2/2 = \nu\alpha_1 t + \frac{\nu}{U_\infty} \, \alpha_2 \, x_1$

and $\quad \eta_1 = y \Big/ \sqrt{2 \, \frac{\nu}{U_\infty} \, (\alpha_2 x_1 + \alpha_1 U_\infty t)}$

The differential equation becomes

$$f_1''' + f_1'' \, (\alpha_1 \eta_1 + \alpha_2 f_1) = 0$$

Consider the special case $\alpha_1 = 1$ and $\alpha_2 = -1$

$$f_1''' + f_1'' \, (\eta_1 - f_1) = 0 \tag{5}$$

where

$$\eta_1 = \frac{y}{\sqrt{2 \, \frac{\alpha\nu}{U_\infty} \, (U_\infty t - x_1)}} \tag{6}$$

This is to be compared with the well-known similar solution for the flat plate case.

$$f_2''' + f_2'' \, f_2 = 0 \tag{7}$$

with

$$\eta_2 = \frac{y}{\sqrt{2 \, \frac{\nu}{U_\infty} \, x_2}}$$

Equation (5) can be transformed into equation (7) by the relations

$$f_2' = 1 - f_1' \quad \text{and} \quad \eta_2 = \eta_1 \tag{8}$$

which in terms of dimensional variables are

$$U_2 = U_\infty - U_1$$

and

$$x_2 = U_\infty t - x_1$$

These are just the transformation equations (1) and (2) considered earlier indicating that the solution to equations (5) and (6) corresponds to the uniformly translating flat plate problem. From this discussion it may be concluded that: (a) As is well known, the velocity field for the uniformly translating flat plate can be considered as the complement to the velocity field for the fixed flat plate in a uniform stream in the sense that $U_2 = U_\infty - U_1$ at a given instant in time; (b) The ordinary differential equation corresponding to the similarity solution for the unsteady velocity field is:

$$f_1''' + f_1''(\eta_1 - f_1) = 0$$

and transforms into the steady case with $f_2' = 1 - f_1'$ and $\eta_2 = \eta_1$ which are just the similarity conditions for the transformation $U_2 = U_\infty - U_1$ and $x_2 = U_\infty t - x_1$; (c) The fact that $f_1'' = -f_2''$ shows that the steady and unsteady problems are dynamically similar (the negative sign occurs because the positive direction of the x coordinate is reversed in the two cases).

III. RELATIONSHIP BETWEEN CONTINUOUS SURFACE AND BOUNDARY LAYER IN A SHOCK TUBE

It is interesting to note that the continuous surface problem also has a counterpart in unsteady flow. If the coordinate system is considered to be attached to the moving belt, the boundary conditions are reversed and the following equation results:

Continuous Surface	Transformation	Unsteady Continuous Surface
$f_2''' + f_2 f_2'' = 0$		$f_1''' + (\eta - f_1) f_1'' = 0$
$f_2(0) = 0\ ,\ f_2'(0) = 1$	$f_1' = 1 - f_2'$	$f_1(0) = 0\ ,\ f_1'(0) = 0$
$f_2'(\infty) = 0$		$f_1'(\infty) = 1$

The continuous surface boundary layer problem is closely related to the boundary layer development in a shock tube. When the coordinate system is fixed with respect to the shock wave and when the shock Mach number goes to infinity (and it would also require the ratio of specific heats, $\gamma = 1$) so that the velocity behind the shock is zero with respect to the wave, we have the steady continuous surface problem as sketched in Figure 3. The unsteady problem refers to the situation where the coordinate system is fixed with respect to the tube. Physically the shock Mach number is not infinite and the boundary layer appears as shown in Figure 4 a situation which has been analyzed by Mirels[4,5]. Mirels, of course, started with the compressible equations which he transformed using the assumptions $\rho\mu$ = constant and Pr = constant. He has also considered approximate solutions and turbulent flow, however, he apparently did not consider the limiting case of $\gamma = 1$ and the $M_s = \infty$.

IV. GENERALIZATION OF BOUNDARY LAYER EQUATIONS

The following development is somewhat different from that of references 4 and 5 in order to achieve slightly more generality. Dynamic similarity suggests that the usual similarity variables should be modified as follows:

$$f'(\eta) = \frac{U - U_\infty}{U_w - U_\infty} \quad , \quad \eta = \frac{y}{g(x)} \tag{9}$$

i.e.,

$$U = U_\infty + (U_w - U_\infty)\, f'(\eta) \tag{10}$$

with boundary conditions $f(0) = 0$, $f'(0) = 1$, $f'(\infty) = 0$. For the sake of generality U_∞ is considered a function of x so that the boundary layer equation with pressure gradient term is required:

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = U_\infty \frac{\partial U_\infty}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2}$$

Note that it is assumed that U_∞ determines the pressure gradient independent of the wall velocity with the result

$$f''' + A f f'' + B \eta f'' - C f'^2 - D f' = 0 \tag{11}$$

where

$$A = \frac{g}{\nu} \frac{\partial}{\partial x} g (U_w - U_\infty) \tag{12a}$$

$$B = \frac{g}{\nu} \frac{\partial}{\partial x} g U_\infty \tag{12b}$$

$$C = \frac{g^2}{\nu} \frac{\partial}{\partial x} (U_w - U_\infty) \tag{12c}$$

$$D = \frac{g^2}{\nu (U_w - U_\infty)} \frac{\partial}{\partial x} U_\infty (U_w - U_\infty) \tag{12d}$$

are the conditions on g and U to insure similarity solutions.

The most general conclusions which can be drawn from these equations are obtained by forming the following combination:

$$\frac{D - B}{2C - A} = \frac{U_\infty}{U_w - U_\infty} = \text{constant}$$

This implies that either (a) U_w is proportional to U_∞, (b) U_w and U_∞ are constant, or (c) $U_w = 0$ or $U_\infty = 0$. Case (a) is the least restrictive condition and the other two cases can be obtained as special conditions

of (a). The dependence of g on $U_w - U_\infty$ and x can be obtained as follows.

First form:

$$A - \frac{C}{2} = \frac{1}{\nu} \frac{\partial}{\partial x} \left(\frac{g^2}{2} (U_w - U_\infty) \right)$$

so that

$$\frac{g^2}{2} = (A - \frac{C}{2}) \nu x/(U_w - U_\infty) \tag{13}$$

This equation can be combined with the C equation to determine the dependence of $U_w - U_\infty$ on x, provided that $A - \frac{C}{2} \neq 0$, i.e.:

$$\frac{C/2}{A - C/2} = \frac{x}{U_w - U_\infty} \frac{\partial}{\partial x} (U_w - U_\infty)$$

so that

$$U_w - U_\infty = C_1 x^{C/(2A-C)} = C_1 x^m \tag{14}$$

and

$$C = \frac{2m}{m+1} A \tag{15a}$$

Remembering that $U_\infty/(U_w - U_\infty)$ is a constant, it is possible also to obtain

$$B = \frac{U_\infty}{U_w - U_\infty} A \tag{15b}$$

$$D = 2 \frac{U_\infty}{U_w - U_\infty} C = \frac{4m}{m+1} \frac{U_\infty}{U_w - U_\infty} A \tag{15c}$$

The differential equation becomes

$$f''' + A \{(ff'' + \frac{U_\infty}{U_w - U_\infty} \eta f'') - \frac{2m}{m+1} (f'^2 + 2 \frac{U_\infty}{U_w - U_\infty} f')\} = 0 \tag{16}$$

For the special case $C = 2A$

$$B = \frac{U_\infty}{U_w - U_\infty} A \tag{17a}$$

$$C = 2A \tag{17b}$$

$$D = 4 \frac{U_\infty}{U_w - U_\infty} A \tag{17c}$$

The differential equation is:

$$f''' + A \{ff' + \frac{U_\infty}{U_w - U_\infty} \eta f'' - 2 (f'^2 + 2 \frac{U_\infty}{U_w - U_\infty} f')\} = 0 \tag{18}$$

which corresponds to the limiting acceleration condition when $m \to \infty$. The similarity variable $\eta = y/g(x)$ has a different form, i.e.,

$$g(x) = \frac{2\nu L}{U_w - U_\infty} \tag{19}$$

where L is a length scale factor and the velocity distribution is

$$U_w - U_\infty = (U_w - U_\infty)_0 \, e^{A \frac{X}{L}} \tag{20a}$$

$$U_\infty = \frac{U_\infty}{U_w - U_\infty} (U_w - U_\infty)_0 \, e^{A \frac{X}{L}} \tag{20b}$$

$$U_w = \left(\frac{U_\infty}{U_w - U_\infty} + 1\right) (U_w - U_\infty)_0 \, e^{A \frac{X}{L}} \tag{20c}$$

where $(U_w - U_\infty)_0$ is a velocity scale factor.

It is usually possible to eliminate one of the constants in equations (16) or (18) by a transformation $f = \alpha f_1$, $\eta = \alpha\eta_1$; however, if the greatest generality is to be retained A must change sign when $U_\infty + U_w$ changes sign although its magnitude can be made arbitrary by the indicated transformation. As long as $U_\infty + U_w > 0$, the boundary layer growth is in the + x direction even though U_∞ may be negative, but when $U_\infty + U_w < 0$ the boundary layer must be considered to grow in the - x direction even if U_w is positive. Figure 5 provides a schematic interpretation of the above concept. If this concept seems too artificial, one can always divide the flow regimes into two problems depending on whether $U_w + U_\infty$ is positive or negative.

Nevertheless, there is some advantage in programming for a computer to writing a single equation encompassing as many boundary conditions as possible; thus, the value of A will be assumed as follows:

$$A = \frac{U_w + U_\infty}{|U_w + U_\infty|} = \frac{2B + 1}{|2B + 1|} \tag{21}$$

The equation for case (a) can now be summarized

$$f''' + \frac{U_w + U_\infty}{|U_w + U_\infty|} \{ff'' + \frac{U_\infty}{U_w - U_\infty} \eta f'' - \frac{2m}{m+1} (f'^2 + 2 \frac{U_\infty}{U_w - U_\infty} f')\} = 0 \tag{22a}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0 \tag{22b}$$

$$g = \sqrt{\frac{2}{m+1} \frac{U_w + U_\infty}{|U_w + U_\infty|} \frac{\nu x}{U_w - U_\infty}} \tag{22c}$$

$$U_\infty = \frac{U_\infty}{U_w - U_\infty} C_1 x^m \tag{22d}$$

$$U_w = \left[\frac{U_\infty}{U_w - U_\infty} + 1 \right] C_1 \, x^m \qquad (22e)$$

The special case (b) where U_w and U_∞ are constant is characterized by $m = 0$. The differential equation reduces to:

$$f''' + \frac{U_w + U_\infty}{|U_w + U_\infty|} \, f \, \left(f + \frac{U_\infty}{U_w - U_\infty} \, \eta \right) = 0 \qquad (23)$$

with only one parameter $U_\infty/(U_w - U_\infty)$. The various flows which are represented by this equation are indicated in Figure 6. The area of the diagram in which U_w and U_∞ are both positive and $U_\infty < U_w$ represents the class of flows considered by Mirels[4,5] and corresponds to the situation of boundary layer growth behind a shock wave. There are two boundaries to the region, namely where there is no relative velocity and where $U_\infty = 0$. The latter case is the continuous surface problem considered by Sakiadis[1]. Below the positive U_w axis the free stream velocity is in the opposite direction to that of the wall, i.e., reverse flow. It is, of course, not obvious that solutions to equation (23) exist in this region. The equation is singular at the point $U_\infty = -U_w$ because $(U_w + U_\infty)/|U_w + U_\infty|$ is discontinuous there. The reverse flow region has a boundary where $U_w = 0$ and this corresponds to the flat plate problem but in the unsteady form in which the boundary conditions are reversed. In the half quadrant where both U_w and U_∞ are negative but $U_\infty < U_w$, we have a slip-flow like regime. It differs from the usual slip-flow problem in that the boundary condition at the wall is a constant velocity whereas in the low density slip-flow problem the wall velocity is assumed proportional to the local velocity gradient and, therefore, a function of x. The slip-flow situation described here was also analyzed by Mirels[5] which he associated with a weak expansion wave. The pressure gradient problem is characterized by $m \neq 0$. The well-known Falkner-Skan flows are obtained when:

$$B = \frac{U_\infty}{U_w - U_\infty} = -1, \qquad A = \frac{U_w + U_\infty}{|U_w + U_\infty|} = -1$$

although the equation is in the transformed coordinates appropriate to the unsteady problem

$$f''' - \{f\, f'' - \eta\, f'' - \frac{2m}{m+1}\ (f'^2 - f')\} = 0 \tag{24}$$

with the boundary condition

$$f(0) = 0, \qquad f'(0) = 1, \qquad f'(\infty) = 0$$

The usual form is recovered through use of the transformation $f' = 1 - f'_2$.

As pointed out, a number of the flow regimes have been explored in papers already in the boundary layer literature; however, by presenting the problem in the generalized way used here, it is possible to identify two regimes, i.e., the reverse flow and the $U_w \propto x^m$ cases which have not been considered previously.

V. SOLUTIONS

Equation 22a has been programmed for the U. S. Army Ballistic Research Laboratories' BRLESC computer using a simple prediction-corrector integration scheme. A step size of .005 in η was used and the integration was carried out to an η of 20. The usual procedure was used of estimating the boundary condition, $f''(0)$, and then correcting the estimate until $f'(20) = 0$ within .00005.

The results of the numerical integration provides f and its derivatives as functions of η. The results are presented graphically in Figures 7 through 15. The boundary layer parameters $f''(0)$, $\bar{\theta}(\infty)$, δ^* and $\bar{\nu}(\infty)$ are given in Table 1.

It is useful to indicate the relationship of the calculated quantities to the conventional boundary layer parameters. Thus, we define the following quantities:

Skin friction coefficient--

$$\frac{c_f}{2} = \frac{\mu \frac{du}{dy}}{\rho\ (U_w - U_\infty)^2} = \frac{g}{x}\ \frac{(m+1)}{2A}\ f''(0) \tag{25}$$

Momentum thickness--

$$\theta = \frac{1}{(U_w - U_\infty)^2} \int_0^\infty U\ (U_\infty - U)\ dy = -g\ \bar{\theta}(\infty) \tag{26}$$

Nondimensional momentum thickness--

$$\bar{\theta}(\infty) = \int_0^\infty \left(\frac{U_\infty}{U_w - U_\infty}\ f' + f'^2 \right) d\eta \tag{27}$$

Displacement thickness--

$$\delta^* = \frac{1}{U_w - U_\infty} \int_0^\infty (U_\infty - U)\ dy = -g\ f(\infty) = -g\ \bar{\delta}^* \tag{28}$$

The integral momentum equation can be used to provide a relationship between $f''(0)$ and $\bar{\theta}(\infty)$ and $f(\infty)$†. The integral momentum equation can be written:

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial x} \int_0^\infty U\ (U_\infty - U)\ dy + \frac{dU_\infty}{dx} \int_0^\infty (U_\infty - U)\ dy$$

Using the above definition for c_f, θ and δ^* this equation becomes:

† Alternatively, equation (27) can be integrated using equation (22a) to eliminate f'^2 with identical results.

$$\frac{c_f}{2} = \frac{d\theta}{dx} + \left(2\theta + \frac{U_\infty}{U_w - U_\infty}\delta^*\right)\frac{1}{U_w - U_\infty}\frac{d(U_w - U_\infty)}{dx} \qquad (29)$$

The differences between this integral equation and its usual form are because of the special definitions used and the fact that $U_\infty/(U_w - U_\infty)$ = constant. Manipulations reduce equation (29) to:

$$f''(0) = -A\left[\frac{1+3m}{1+m}\bar{\theta}(\infty) + \frac{U_\infty}{U_w - U_\infty}\frac{2m}{m+1}f(\infty)\right] \qquad (30)$$

When m = 0 there is a particularly simple relationship between f (0) and θ and the following well known result is obtained:

$$c_f = \frac{\theta}{x} \qquad (31)$$

The integral $\bar{\theta}(\infty)$ was also numerically calculated with each integration of the basic equation.

In addition to the characteristics indicated above, it is of interest to calculate the y component of velocity, V, since it is an important indicator of the behavior of the flow. It is obtained by integration of the continuity equation:

$$V = -\int_0^y \frac{\partial U}{\partial x}\,dy, \qquad V\,(y=0) = 0$$

$$V = -g\,\frac{d(U_w - U_\infty)}{dx}\left[\eta\,\frac{U_\infty}{U_w - U_\infty} + f\right] + (U_w - U_\infty)\frac{dg}{dx}\,[\eta f' - f] \qquad (32)$$

which can be reduced to:

$$\bar{V} = \left(\frac{2}{m+1}\,\frac{V}{U_w - U_\infty}\,\frac{Ax}{g}\right) = -A\,\left\{f + \frac{2m}{m+1}\,\frac{U_\infty}{U_w - U_\infty}\,\eta - \frac{1-m}{1+m}\eta f'\right\} \qquad (33)$$

For large values of η, this equation goes to the following limiting form:

$$\lim_{\eta \to \infty} V = -A \{ f(\infty) + \frac{2m}{m+1} \frac{U_\infty}{U_w - U_\infty} \eta \} \tag{34}$$

This equation combined with the $U = cx^m$ satisfy the continuity equation and may be considered the inviscid free stream velocity components

$$U_\infty = c_1 x^m$$

$$V_\infty = -A \{ f(\infty) + \frac{2m}{m+1} \frac{U_\infty}{U_w - U_\infty} \frac{y}{g(x)} \}$$

Figure 7 shows the nondimensional shear stress at the wall, $-f''(0)$ as a function of $B = U_\infty/U_w - U_\infty$ with $\beta = 2m/m+1$ as a parameter. The $m = 0$ curves show that $-f''(0)$ decreases as the parameter B approaches the reverse flow regime from above and from below. However, the two sections of the curve are not symmetric about the singular point $B = -1/2$. A few tentative calculations were made in the reverse flow region for $m = 0$ and they suggest a continuous decrease in $-f''(0)$, approaching zero near $B = -1/2$. However, it is not clear that these calculations represent real solutions because of the limited range of integration. In fact, when solutions even closer to $B = -1/2$ were attempted, the calculations did not converge. Figures 8 and 9 show the x and y velocity components (nondimensional) for $\beta = 0$ and for $B \geq 0$ and Figures 12 and 13 show the corresponding velocities for the $B < -1/2$ region.

Only a few preliminary points were calculated for the $\beta \neq 0$ case. Almost all these calculations were made for the $\beta = .25$ ($m = 1/7$) or $\beta = -.1$ ($m = -1/4$) cases. The shear stress $-f''(0)$ is higher for the $\beta = .25$ situation shown in Figure 7 and lower for the $\beta = -.1$. The velocity distributions are shown in Figures 10 and 11 for $B = +1$ and in Figures 14 and 15 for the $B = -1$. For comparison some of the Falkner-Skan solutions corresponding to $B = -1$ and $-.199 < \beta < 2$ are shown in Figures 14 and 15. Although they are not shown here, the Falkner-Skan

equations in the adverse pressure gradient case are non-unique; the second branch solutions which are not shown also have a reverse flow character with positive $f''(0)$ and a region of the flow near the wall moving in a direction opposite to that of the main stream. The reverse flows described by $-1 < \beta < 0$ in this report are caused by the wall moving in a direction opposite to the free stream while the second branch Falkner-Skan solutions have $U_w = 0$. The second branch Falkner-Skan are also a direct continuation of the adverse pressure gradient solutions beyond the "separation" condition corresponding to $f''(0) = 0$ with $\beta = -.199$. All the second branch solutions as well as the present reverse flow solutions cannot be assumed to represent physically real flows. To be realized, they must refer to a finite body with effectively two leading edges, one at $x = 0$ and the other down stream. These solutions perhaps apply in an intermediate region with a region around the second leading edge omitted from consideration just as the region around the forward leading edge must be omitted because the boundary layer equations do not apply there (as they also do not apply at the separation point). This interpretation is quite similar to the suggestion made by Libby in interpretating the overshoot profile solutions which were recently found for values of $\beta < -.199$ (reference 12). No attempt has been made experimentally to confirm or refute this speculation, however, because of the difficulties in producing such flows. The reverse flow situation with a continuous belt might prove somewhat easier to investigate experimentally and thus might contribute to our understanding of similar solutions.

The $\beta \neq 0$, non-reverse flow solutions may be interpreted as boundary layer solutions behind accelerating shock waves with the assumption that compressibility effects can be eliminated by a suitable transformation or idealization of the fluid ($\rho\mu$ = constant) and a restriction on the pressure gradient condition in order to preserve the incompressible fluid requirement.

VI. SUMMARY

By introducing similarly variables in velocity and normal coordinate of the form,

$$f'(\eta) = \frac{U - U_\infty}{U_w - U_\infty}$$

$$\eta = y/g(x) = y\sqrt{\frac{m+1}{2}\,\frac{(U_w + U_\infty)}{|U_w + U_\infty|}\,\frac{U_w - U_\infty}{\nu x}}$$

the following general differential equation can be derived from the incompressible laminar boundary layer equations:

$$f''' + \frac{U_w + U_\infty}{|U_w + U_\infty|}\left((f''f + \frac{U_\infty}{U_w - U_\infty} f''\eta) - \frac{2m}{m+1}(f'^2 + 2\frac{U_\infty}{U_w - U_\infty} f')\right) = 0$$

with boundary conditions $f(0) = 0$, $f'(0) = 1$, $f'(\infty) = 0$ and $U_\infty \propto U_w \propto x^m$. This equation encompasses as special cases, the following well-known similar solutions:

	$U_\infty/(U_w - U_\infty)$	β
Blasius (Flat Plate)	-1	0
Falkner-Skan (Accelerating Flow on a Flat Plate)	-1	-.199 << 2
Mirels (Boundary Layer Behind a Shock or Expansion Wave)	-1 >> -∞ .2 << ∞	0
Sakiadis (Continuous Surface)	0	0
Moore (Accelerated Continuous Surface)	-3/2, -10/9	<0

Formulating the problem in this way shows that in addition to the solutions already obtained there are a number of ranges of the parameters $U_\infty/(U_w-U_\infty)$ and β which have not previously been investigated. Two of these regions are:

$$\text{(a)} \quad U_w \propto x^m, \quad U_\infty = 0$$

$$\text{(b)} \quad -1 < U_\infty/U_w - U_\infty < 0$$

(reverse flow)

A number of numerical solutions have been computed and are compared with the well-known solutions.

Table 1. Boundary Layer Parameters

B	β	$f''(0)$	$\bar{\theta}$	$\bar{\delta}^* = f(\infty)$	$\bar{V}(\infty)$	$H = \bar{\delta}^*/\bar{\theta}$	δ	Comments
				$\beta = 0$				
4.0	0	-1.718	1.718	.375	- .375	.218	1.36	
3.0	0	-1.521	1.520	.424	- .425	.279	1.55	
2.0	0	-1.295	1.294	.502	- .502	.388	1.83	
1.0	0	-1.020	1.019	.647	- .647	.635	2.42	
0.333	0	- .782	.782	.868	- .868	1.110	--	Ref. 5
0.20	0	- .725	.725	.950	- .950	1.310	--	Ref. 5
0.00	0	- .626	.630	1.164	-1.163	1.848	5.84	Continuous Surface
-0.10	0	- .561	.566	1.682	-1.666	2.972	18.51	
-0.50								No Solutions
-0.80	0	- .287	- .288	1.788	1.788	-6.208	4.97	
-0.90	0	- .392	- .393	1.412	1.411	-3.593	4.20	
-1.0	0	- .4695	- .470	1.218	1.218	-2.591	3.75	Flat Plate
-1.333	0	- .661	- .661	.905	.905	-1.369	--	Ref. 5
-2.0	0	- .930	- .930	.662	.662	- .712	2.23	
-3.0	0	-1.226	-1.226	.509	.509	- .415	1.76	
-4.0	0	-1.463	-1.463	.429	.429	- .293	1.50	
				$\beta \neq 0$				
2.0	0.25	-1.655	1.145	.447		.390	1.75	
1.0	.25	-1.282	.909	.583		.641	2.33	
1.0	.50	-1.507	.826	.535		.648	2.23	
1.0	- .10	- .900	1.075	.678		.631	2.47	
0.0	.25	- .733	--	1.117		--	5.98	
0.0	- .10	- .581	.648	1.178		1.818	5.78	
-0.5								No Solutions

Table 1. Boundary Layer Parameters (Continued)

B	β	$f''(0)$	$\bar{\theta}$	$\bar{\delta}^*=f(\infty)$	$\bar{V}(\infty)$	$H=\bar{\delta}^*/\bar{\theta}$	δ	Comments
				$\beta \neq 0$				
-1.0	- .10	- .319	- .515	1.444		-2.804	4.03	
-1.0	- .199	0.0	- .576	2.349		-4.078	4.99	
-1.0	.25	- .732	- .396	.945		-2.386	3.29	Falkner-
-1.0	.50	- .928	- .350	.804		-2.297	2.99	Skan Solutions
-1.0	1.0	-1.233	- .292	.649		-2.223	2.63	
-1.0	2.0	-1.687	--	--		--	2.23	
-2.0	- .10	- .758	-1.004	.723		- .720	2.32	
-2.0	.25	-1.279	- .799	.559		- .700	2.06	
-3.0	.25	-1.652	-1.059	.436		- .399	1.62	

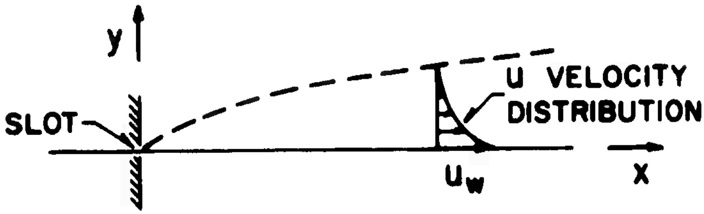

Figure 1. Continuous Surface Boundary Layer

Figure 2. Coordinates for a Moving and Fixed Flat Plate

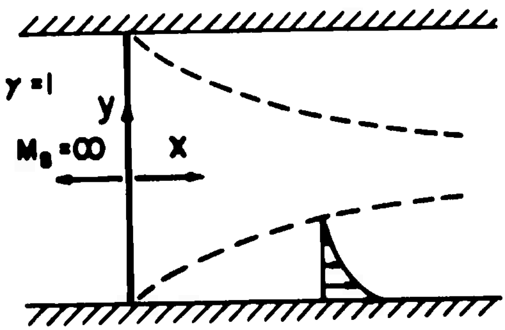

Figure 3. Continuous Surface Boundary Layer as Limiting Condition of Shock Tube Boundary Layer ($M_s = \infty$, $\gamma = 1$)

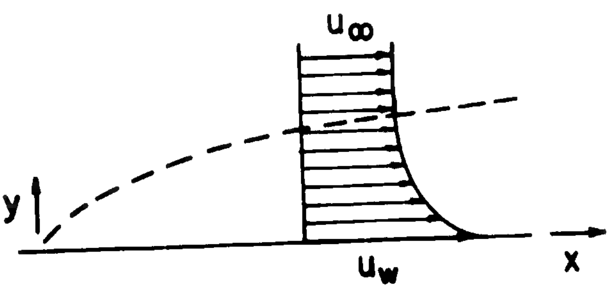

Figure 4. Shock Tube Boundary Layer ($M_s < \infty$)

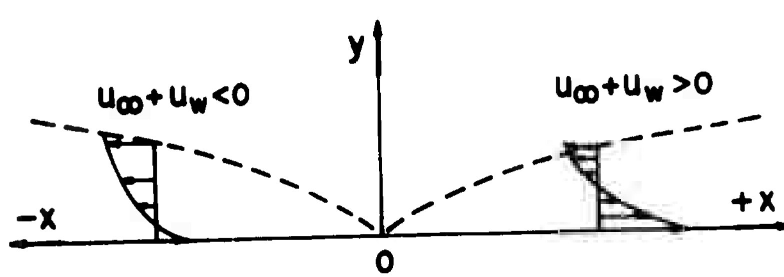

Figure 5. Direction of Boundary Layer Growth

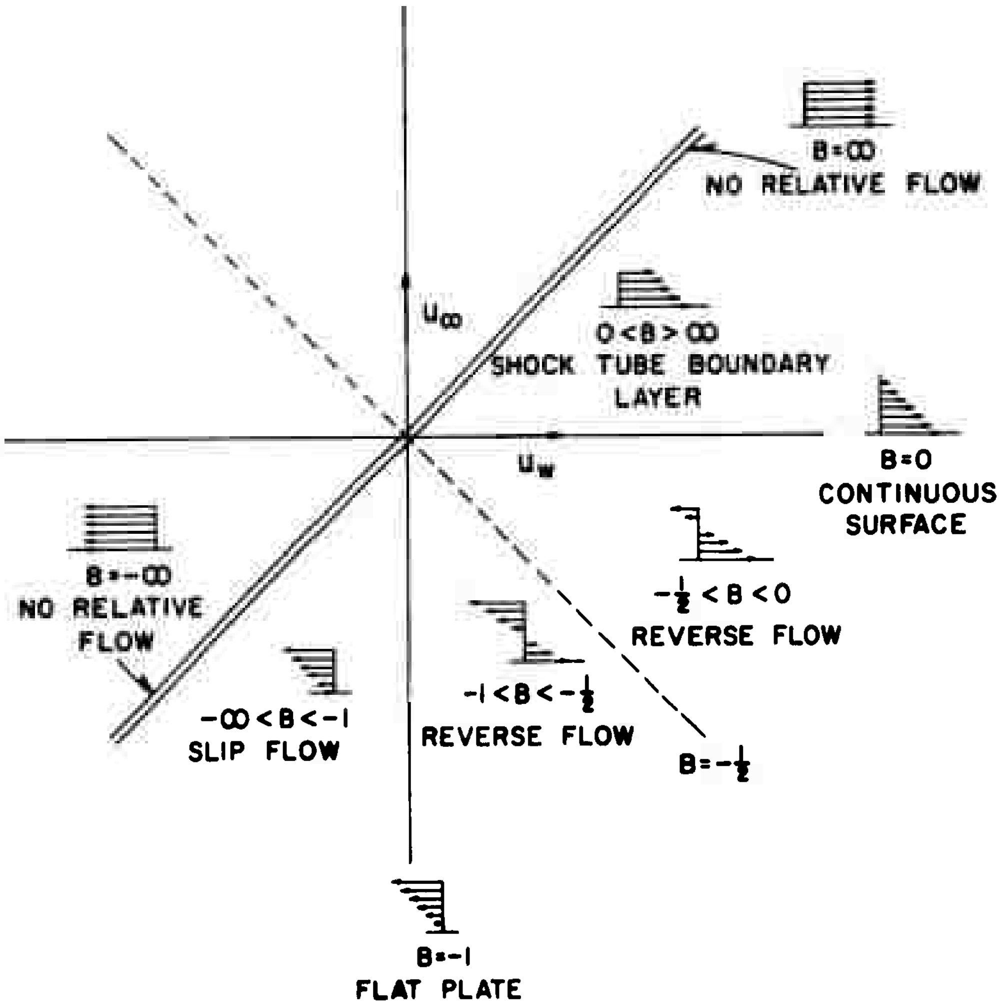

Figure 6. Flows Represented by $f''' + \frac{2B+1}{|2B+1|} f'' (f + B\eta) = 0$

where $B = U_\infty/(U_W - U_\infty)$

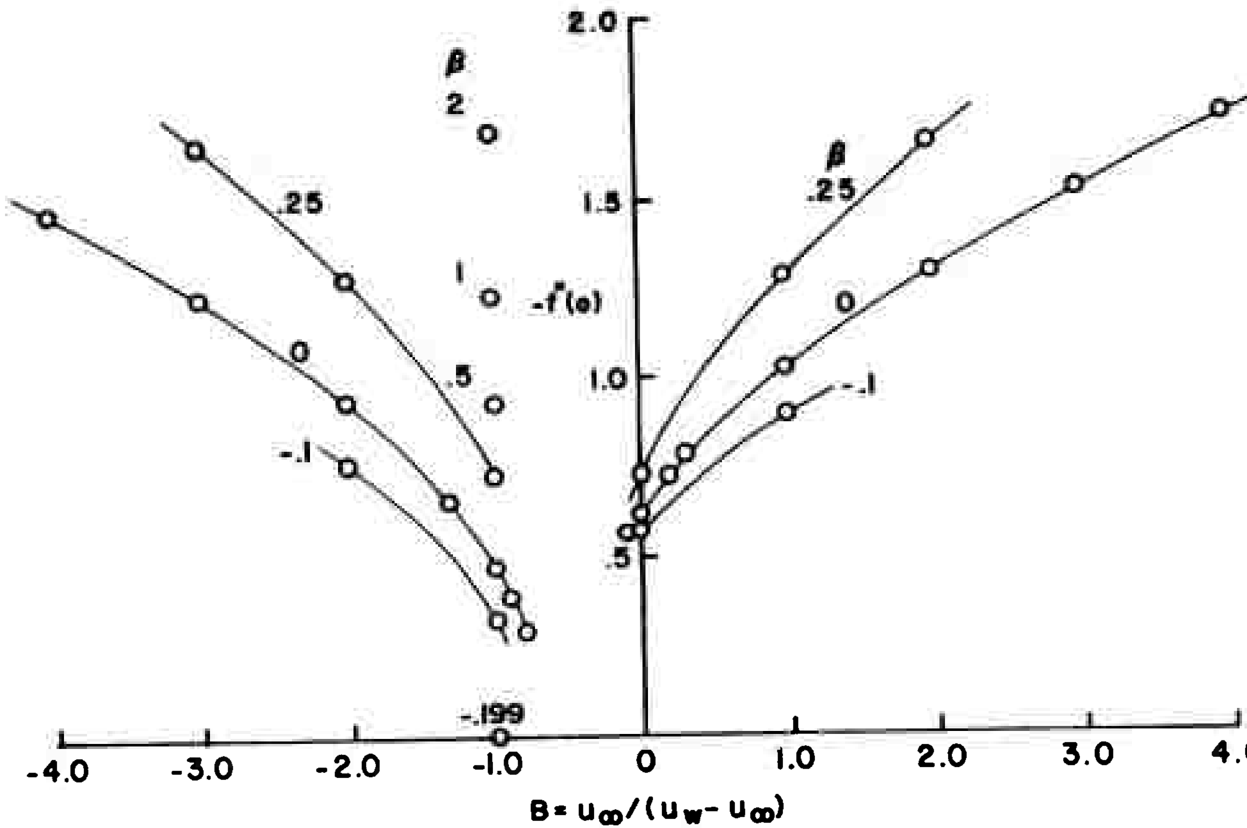

Figure 7. Nondimensional Wall Friction, $f'(0)$ Versus B

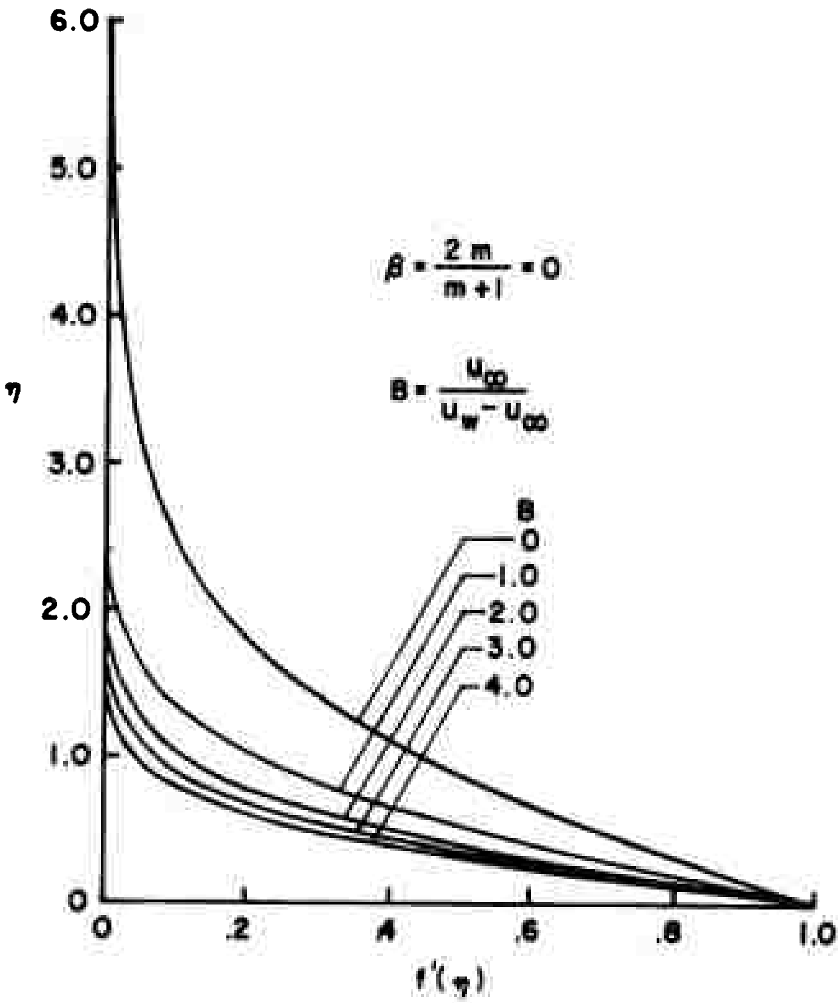

Figure 8. f′ Versus η, B ≥ 0, β = 0

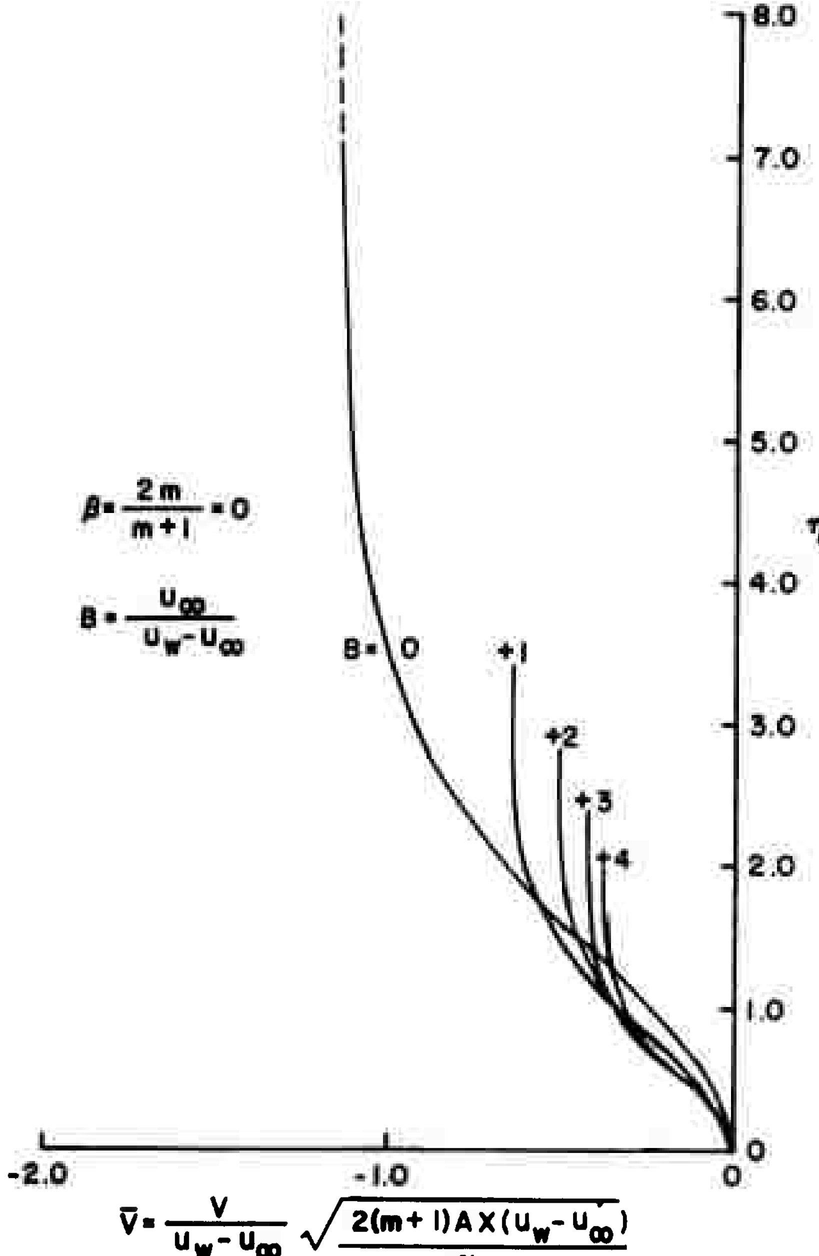

Figure 9. $\bar{V}$ Versus η, $B \geq 0$, $\beta = 0$

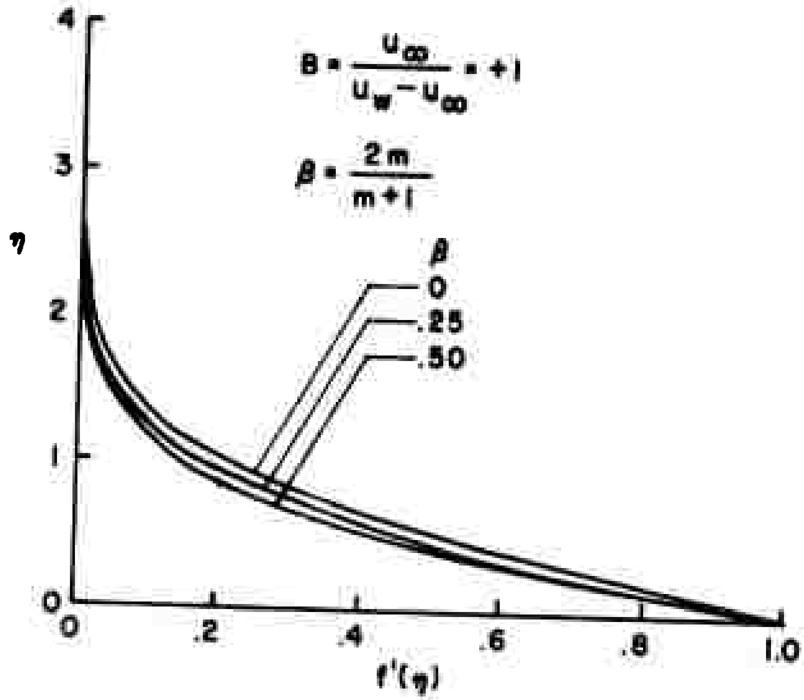

Figure 10. f' Versus η, $B = +1$, $\beta = 0, .25, .50$

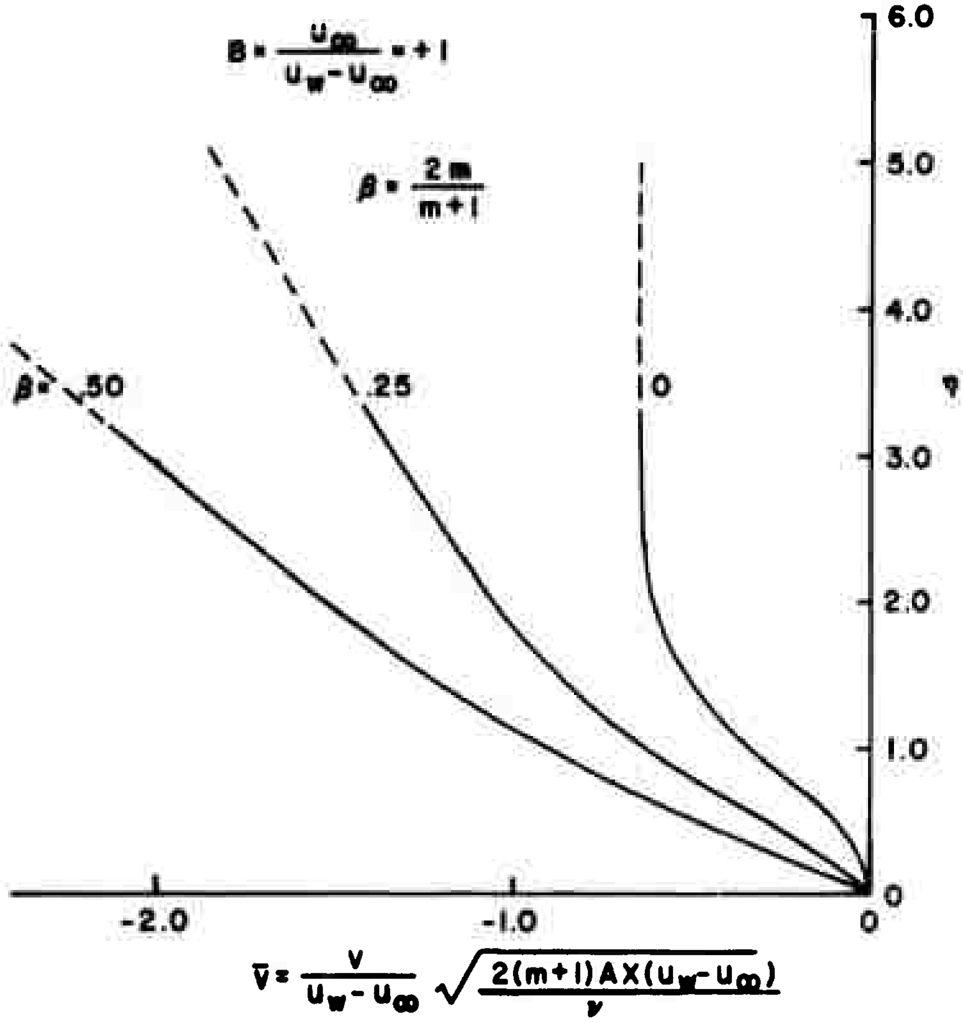

Figure 11. $\bar{V}$ Versus η, $B = +1$, $\beta = 0, .25, .50$

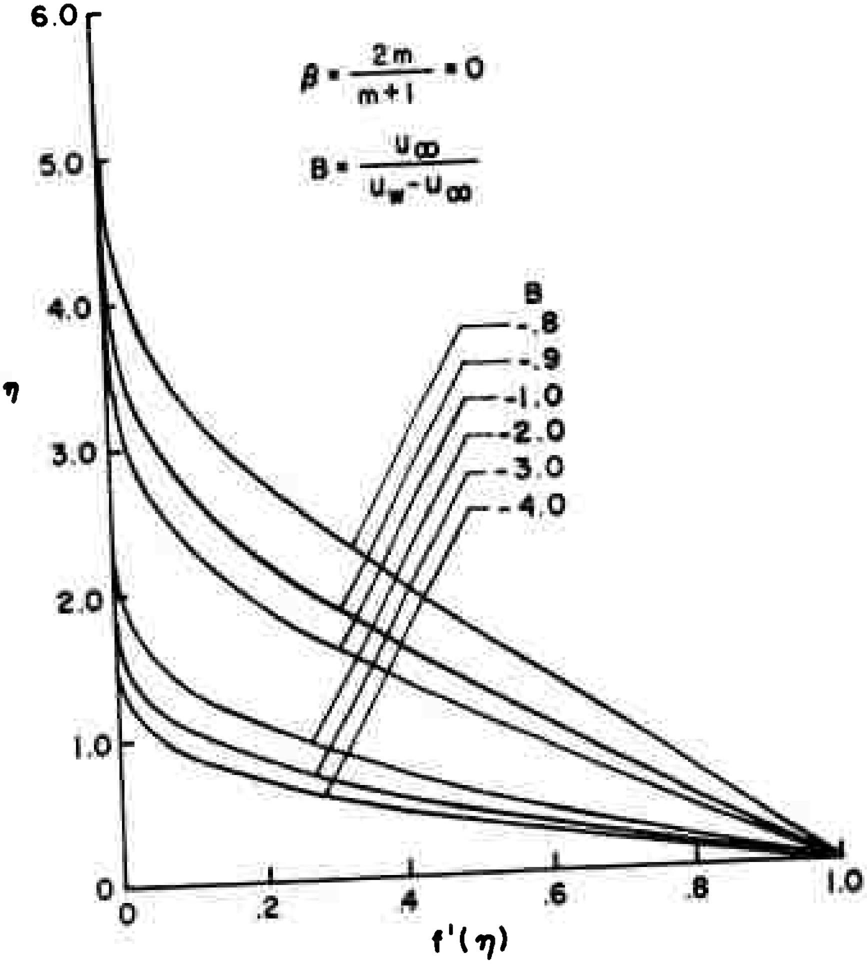

Figure 12. f' Versus η, $B < -1/2$, $\beta = 0$

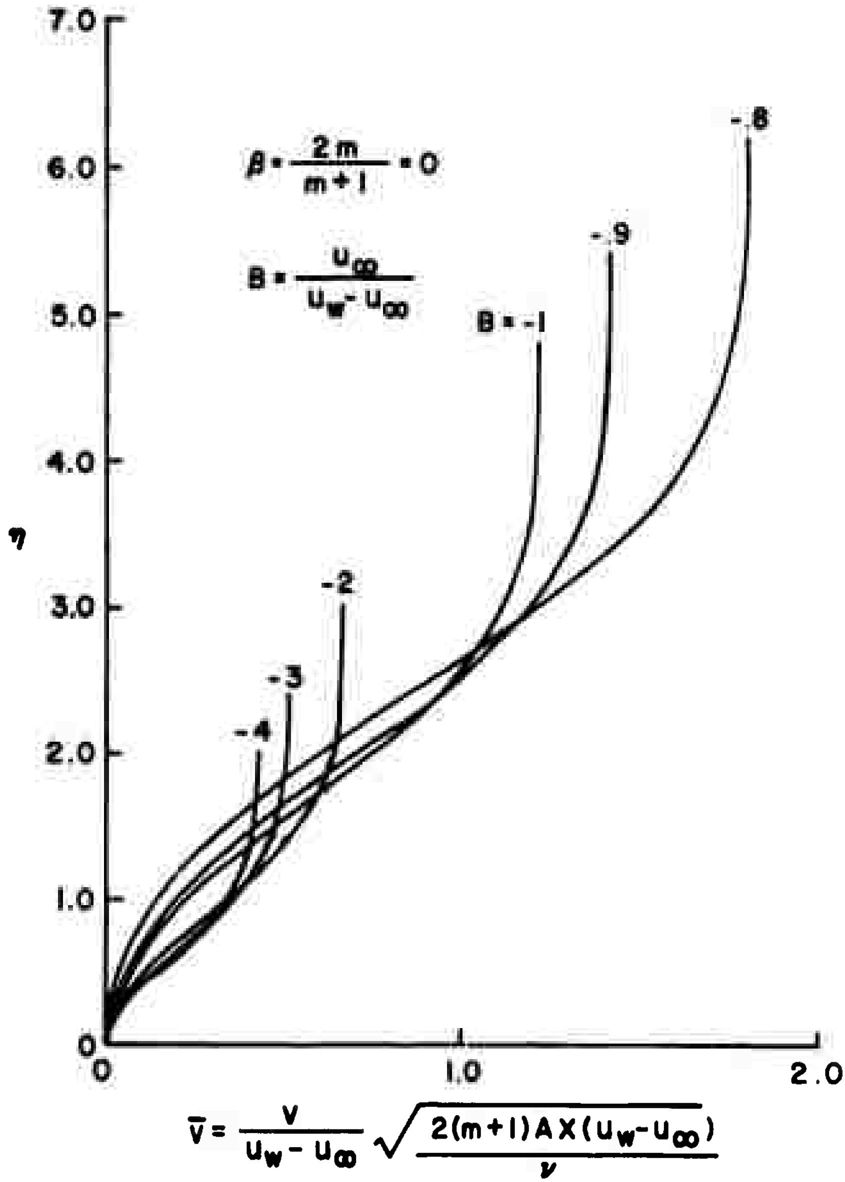

Figure 13. $\bar{V}$ Versus η, $B < -1/2$, $\beta = 0$

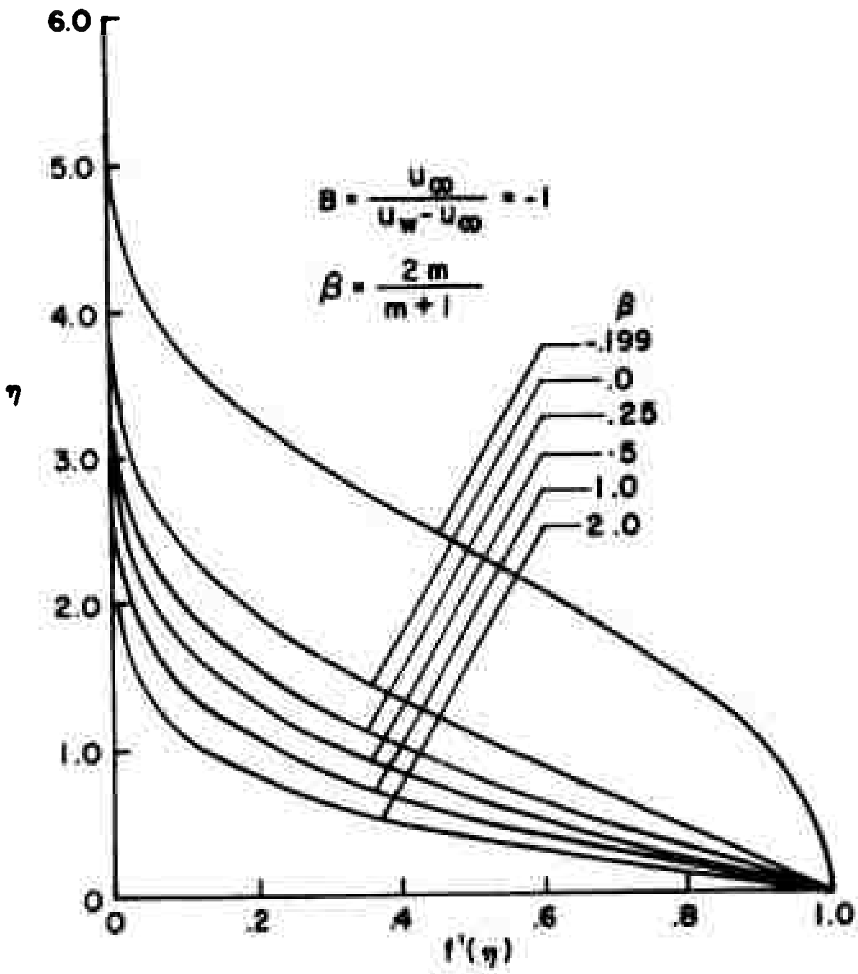

Figure 14. f' Versus η, $B = -1$, $-.199 \leq \beta \leq 2$ (Falkner-Skan Solution)

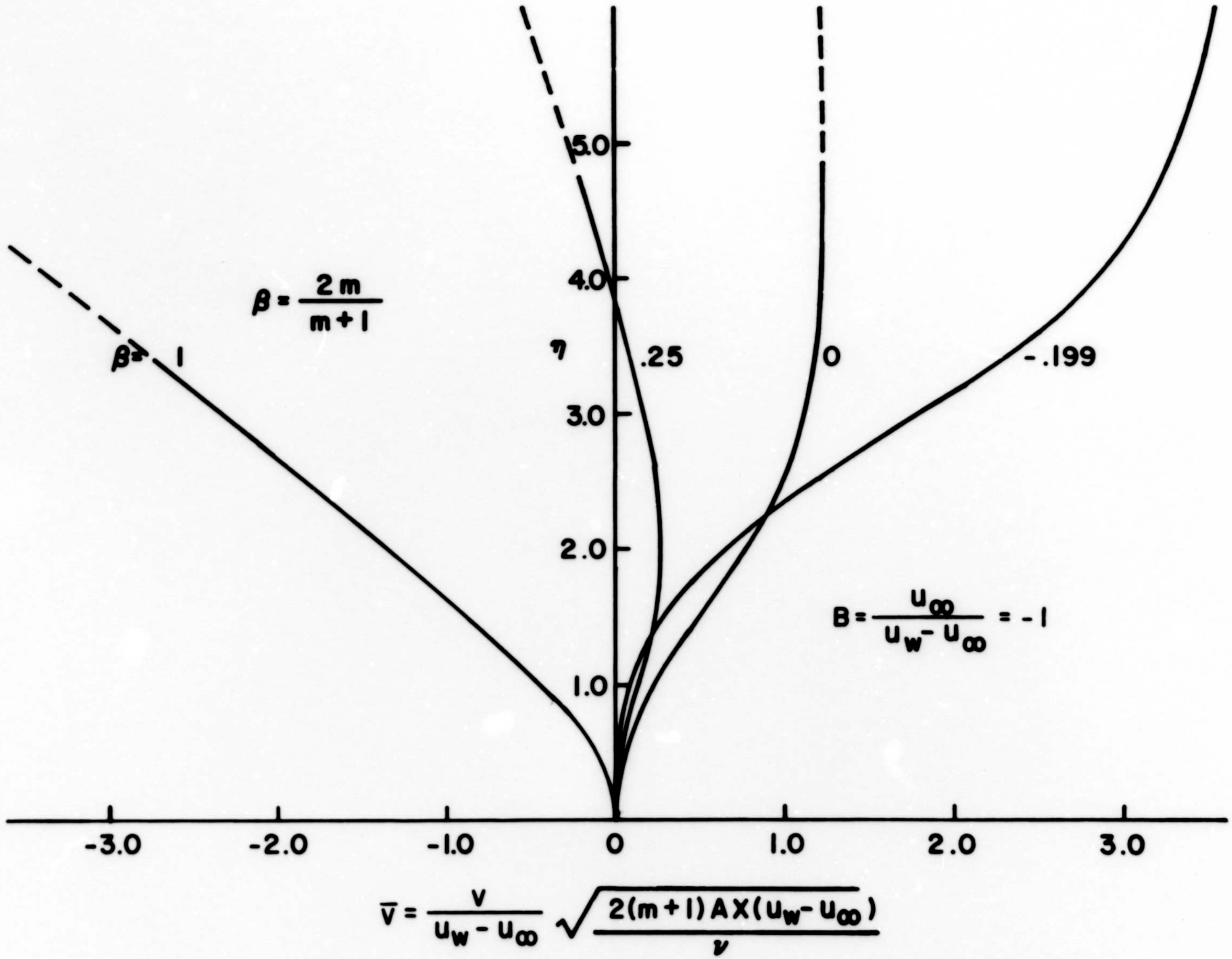

Figure 15. V Versus η, B = -1, -.199 ≤ β ≤ 2 (Falkner-Skan Solution)

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

U. S. Army Aberdeen Research & Development Center
Ballistic Research Laboratories
Aberdeen Proving Ground, Maryland 21005

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

GENERALIZATION OF THE SIMILARITY EQUATIONS FOR CONTINUOUS SURFACE BOUNDARY LAYERS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

James E. Danberg

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
October 1970	51	14

8a. CONTRACT OR GRANT NO.	**9a. ORIGINATOR'S REPORT NUMBER(S)**
b. PROJECT NO. RDT&E 1T061102A33D	BRL Report No. 1504
c.	**9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)**
d.	

10. DISTRIBUTION STATEMENT

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U. S. Army Materiel Command Washington, D. C.

13. ABSTRACT

Through a generalization of the continuous surface problem (i.e., boundary layers on a moving belt or behind a shock wave), an equation is derived which encompasses many of the well known similarity solutions of boundary layer theory as well as some solutions not previously considered. Since these problems involve both free stream and wall velocities, similarity variables are introduced which depend on velocity differences. The parameter $B = U_\infty/(U_w - U_\infty)$ describes the relative importance of the boundary conditions along with the usual pressure gradient parameter, β. This formulation of the problem includes the following special cases: flat plate (Blasius), accelerating or decelerating flow (Falkner-Skan), boundary layer behind a shock or expansion wave (Mirels), continuous surface (Sakiadis) and accelerating wall and free stream (Moore). Two new conditions not previously considered involve reverse flow and acceleration and deceleration of a continuous surface boundary layer. Preliminary numerical calculations have been made for these conditions.

B = U sub infinity/(U sub W - U sub infinity)

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Laminar Boundary Layer Continuous Surface Similar Solutions						